Production function and short/long-run average costs

A firm has a production function $q = 2k^{1/2}l^{1/2}$.

1. What type of returns to scale does this production function have?
2. Suppose that k is fixed at a value $\bar{k}$ in the short run. Find the total cost function in the short run and the short-run average cost function.
3. What must be the relationship between the short-run capital level $\bar{k}$ and q to minimize costs of producing q?
4. Use your answer from the previous point to find the long-run total cost function.
5. Write the short-run and long-run total and average cost functions for $r = \$1$, $w = \$4$.
6. Prove that the minimum short-run average cost is given by $q = \bar{k}$.
7. Prove that the minimum short-run average cost coincides with the minimum long-run average cost for all q.

Answers

1.

$$F(mk, ml) = 2(mk)^{0.5}(ml)^{0.5} = m2k^{1/2}l^{1/2} = mf(k, l) = mq$$

Constant returns to scale.

2.

$$\min_{L} \; r\bar{k} + wL$$

subject to the constraint:

$$q = f(\bar{k}, L)$$

For instance, if the production function is $q = 2\sqrt{\bar{k}L}$, the minimization problem becomes:

$$\min_{L} \; r\bar{k} + wL$$

subject to:

$$q = 2\sqrt{\bar{k}L}$$

Solve the Constraint for L

From the constraint $q = 2\sqrt{\bar{k}L}$, we can solve for L:

$$L = \frac{q^2}{4\bar{k}}$$

Substitute L into the objective function:

$$c(q) = r\bar{k} + w\left(\frac{q^2}{4\bar{k}}\right)$$

$$c(q) = r\bar{k} + \frac{wq^2}{4\bar{k}}$$

$$\text{average cost } : c/q = \frac{r\bar{k}}{q} + \frac{wq}{4\bar{k}}$$

3.

$$\bar{k} \text{ that minimizes } C:$$

$$\frac{\partial C}{\partial \bar{k}} = r - \frac{wq^2}{4\bar{k}^2} = 0 \rightarrow \bar{k}^2 = wq^2/(4r)$$

$$\bar{k}^* = \sqrt{\frac{w}{r}} \cdot \frac{q}{2}$$

4.

$$C(q) = r\left[\sqrt{\frac{w}{r}} \cdot \frac{q}{2}\right] + \frac{wq^2}{4\left(\sqrt{\frac{w}{r}} \cdot \frac{q}{2}\right)}$$

$$C(q) = \sqrt{r}\left[\sqrt{w} \cdot \frac{q}{2}\right] + \sqrt{rw}\frac{q}{2}$$

$$C(q) = \sqrt{rw} \cdot q$$

5.

$$C = \bar{k} + \frac{q^2}{\bar{k}}$$

$$\text{short-run average cost: } = \frac{\bar{k}}{q} + \frac{q}{\bar{k}}$$

$$CT(q) = 2q$$

$$\text{long-run average cost: } = 2$$

6.

$$\frac{\partial \text{short-run average cost}}{\partial q} = -\frac{\bar{k}}{q^2} + \frac{1}{\bar{k}} = 0 \rightarrow q = \bar{k}$$

7.

$$\text{short-run average cost} = \frac{\bar{k}}{\bar{k}} + \frac{\bar{k}}{\bar{k}} = 2$$

which is the value of the (constant) long-run average cost found in point (5).